A Transformational Approach to Factoring

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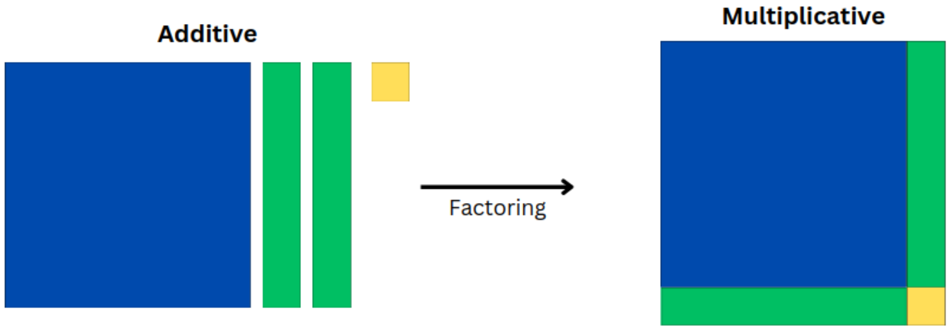
Abstract
Factoring nonmonic quadratics is challenging. To overcome such difficulties, we offer a substitution method that transforms challenging expressions into a more familiar structure. This approach illustrates a common problem-solving strategy and engages students in the mathematical practice of looking for and using structure.

Keywords: Factoring quadratics, mathematical practices, mathematical structure, problem solving

1 Introduction

As the Common Core State Standards (2010) highlight, a fundamental algebraic understanding is the ability to "produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression." One topic that embodies such an understanding is factoring. Factoring is the process of transforming an algebraic expression from an additive structure to a multiplicative structure. An expression's multiplicative structure is a powerful form, revealing zeros of a polynomial and common factors which aid in simplifying and combining rational polynomials as well as locating local optima of functions. Given that the factored form is the multiplication of two (or more) expressions, these factors can be interpreted as the dimensions of a rectangle with the resulting product the area (or volume, etc.).

Figure 1: Geometric Representation of Additive and Multiplicative Forms of $x^2 + 2x + 1$.



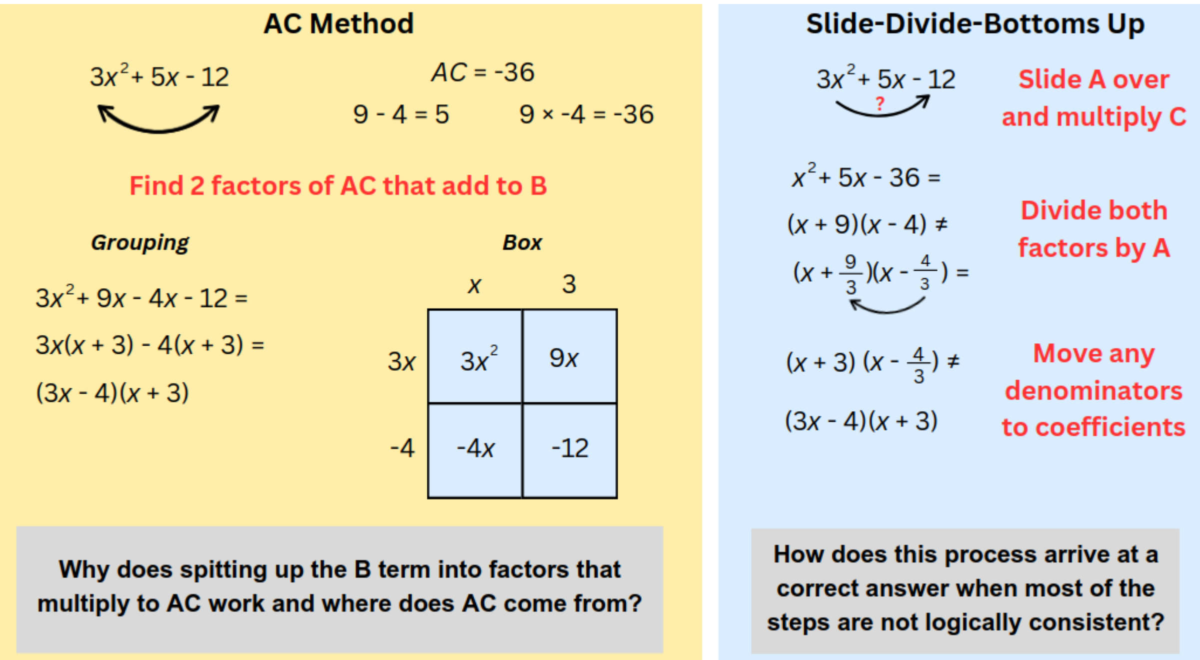
However, while factoring dominates the curriculum in the United States, students continue to struggle to fluently carry out this process (Kotsopoulos, 2007). No doubt, learning to meaningfully factor an expression can be challenging. In most cases the process cannot be determined directly, but rather requires reversing the distributive property by observing patterns that result in specific forms and then undoing this process. Such a method is often referred to as guess and check. For example, when factoring trinomials of the form $ax^2 + b^x + c$, students might repeatedly multiply two binomials, $(x+m)(x+n)$ and notice that when $a = 1$ the c

term results from the multiplication of the two factors ($c = m \times n$) and the b term is their sum ($b = m + n$). Such pattern recognition is at the heart of the mathematical practice Looking for and Using Structure (CCSS, 2010). However, as all secondary teachers know, while this connection is quite discernable for monic quadratics where $a = 1$, it is considerably more challenging to understand and apply when $a \neq 1$.

To explore how teachers support students in overcoming such a challenge, we interviewed multiple high school teachers, representing a diverse range of schools and experience (Prins & Hawthorne, 2024). Overwhelmingly, we found that while most teachers engaged students tracked in honors classes in a similar guess and check method, they almost all avoided more conceptually grounded approaches with lower-tracked students and instead presented them with various black box algorithms. As illustrated below, two of the more popular algorithms, the AC Method (involving both grouping and box scaffolding) and Slide-Divide-Bottoms Up (also referred to as Slip & Slide), both consist of multiple steps that go unexplained (Steckroth, 2015). Not only does such an algorithm promote a rote view of mathematics, both procedures involve moving and manipulating isolated symbols, encouraging students to treat algebraic expressions as disconnected strings of objects rather than a meaningful representation.

We recognize that factoring non-monic trinomials is difficult. However, rather than presenting black box algorithms, we argue that teachers should leverage this opportunity to model how mathematicians overcome such difficulties. One common technique is to find a way to transform the problem into an easier or familiar problem. While factoring might seem like a mindless procedure, once reframed around a rich problem solving strategy, we believe it can provide students an opportunity to develop key mathematical ways of thinking. With such a goal in mind, we explain what a transformational approach to problem solving entails and how it applies to factoring, as well as provide an instructional trajectory to aid teachers in implementing this approach in the classroom.

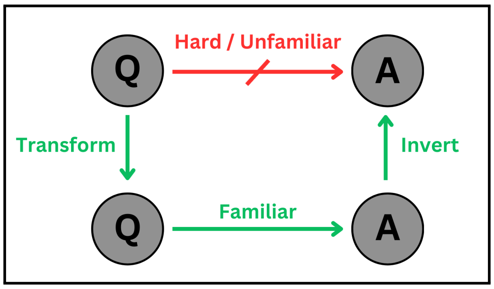
Figure 2: AC and Slide-Divide-Bottoms-Up methods for factoring. Non-equivalent steps are highlighted in red.



2 Transformations

When faced with a problem where the solution is unknown or complicated, a common strategy employed by mathematicians is to look for ways to transform the question into an alternative form with an easier and known solution path. Figure 3 illustrates this thought process, which involves converting to an easier form, solving this newly created problem, and then "inverting" the answer back to the original form. One example of this approach is the u-substitution method used in integral calculus. While a problem such as $\int \cos(x)\sin^3(x)\,dx$ looks quite daunting, by assigning $u = sinx$ and $du = cosx$, it can be transformed into a polynomial $\int u^3\,du$. Now in a familiar form, it can easily be integrated using the power rule, yielding a result of $\frac{u^4}{4}+c$. From there we can invert back by substituting $sinx$ in for u, arriving at the solution of $\frac{sin^4(x)}{4}+c$. This process of transforming a mathematical object or expression into something familiar then inverting back shows up in many areas of mathematics such as solving systems of equations and matrix transformations.

Figure 3: Transforming problems to easier, familiar methods.

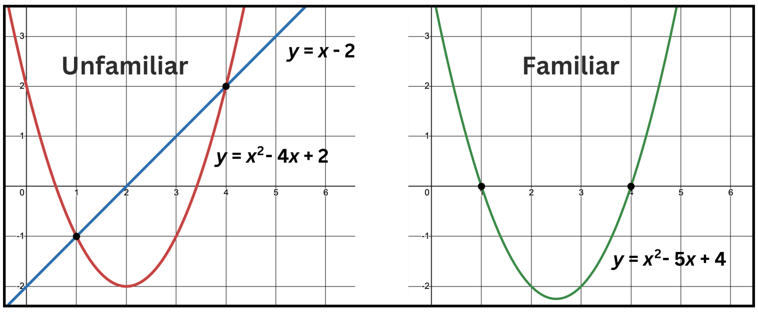


3 Look For and Make Use of Structure

While such techniques are often taught as rules, the ability to see and productively leverage transformations involves much more than carrying out a procedure. It requires focused practice to develop the mathematical habit to look for and make use of structure, something often missing in our students. For example, Hoch & Dreyfus (2004) found that almost 90% of college bound juniors, when asked to solve $\frac{1}{4} - \frac{x}{x-1} - x = 6 + \frac{1}{4} - \frac{x}{x-1}$, began by finding a common denominator or cross multiplying, rather than observing that the expression $\frac{1}{4} - \frac{x}{x-1}$ is common to both sides of the equation. Clearly such a connection is not obvious, requiring a specialized way of thinking to see and leverage the structure at hand. In the previous calculus example, students have to see $cosx(sinx)^3$, not as a string of symbols, but as the product of two functions, the latter being a composition of functions and the former being the derivative of the inside function.

Notably, while structure is often viewed and transformed symbolically, mathematicians also look to other representations. For example, leveraging the underlying structure to solve the inequality $x^2-4x + 2 < x-2$ is quite complicated. However, by graphing the two functions involved in the inequality, the question can be viewed as finding the x-values where the linear function is greater than the quadratic function (See Figure 4). Furthermore, one can reconceptualize the problem by imagining a new function, the difference in the two functions, which further transforms the question to finding the x-values which produce positive values in the new function. Again, while many students might be supported in an algebraic approach of moving symbols around, combining like terms and factoring, teaching this through a transformational approach develops a critical mathematical practice as well as leads to deeper understanding.

Figure 4: Transforming inequalities by conceptualizing a single, graphical function.



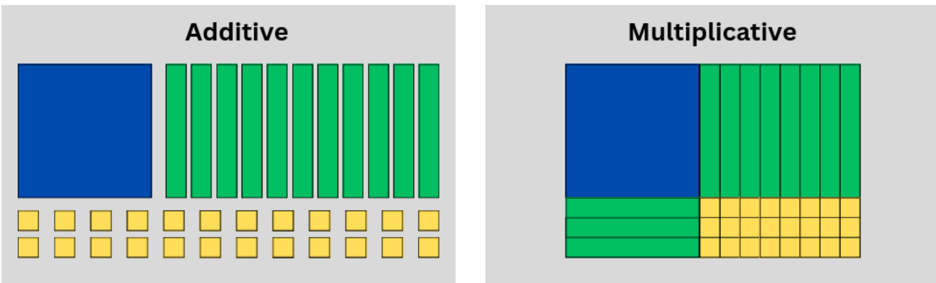
4 Transformational Approach to Factoring

To overcome the difficulties associated with factoring non-monic quadratics, we offer a transformational approach which involves converting to a more familiar structure involving a monic quadratic. Notably, this method not only simplifies the factoring process, but also explains many of the black box algorithms (Prins, et al., in review). To better demonstrate the underlying structure at the heart of such a transformation, we will use an area model, building off the work of Lischka and Stephens (2020), and illustrate each phase of the process using algebra tiles. However, before walking the reader through the transformation, we begin by showing how algebra tiles can be used to identify and generalize patterns involved in factoring of the more intuitive case involving monic quadratics.

4.1 Factoring Quadratics when $a = 1$: The Familiar Case

To factor with algebra tiles, we start with a collection of different sized shapes representing a trinomial in expanded or additive form and arrange them into a rectangle whose dimensions form the product of two expressions—the multiplicative or factored form. For example, the expression $x^2 + 11x + 24$ consists of a single $x \times x$ tile (or square), 11 $x \times 1$ tiles (planks or "longs") and 24 1×1 tiles (or units). To arrange these pieces in a rectangle, we start by placing the square in the top left and recognize that the 11 planks must be distributed on adjacent sides because the square and the planks share the one x dimension. In each case the placement of the 11 planks will leave a rectangle, to be filled in with the remaining units, whose dimensions will be dictated by the exact distribution of the 11 planks. Since this rectangle must consist of 24 units, the goal is to find a distribution of the planks which leave a rectangle with an area of 24. Looking at the factors of 24, there are four ways to arrange the units into a rectangle, but only one that aligns with exactly 11 planks. Symbolically, we need to find two numbers (dimensions of the rectangle) l and w, such that $l \times w = 24$ and $l + w = 11$. The correct configuration for this problem would be 3×8 (See Figure 5).

Figure 5: Collection of algebra tiles representing/equivalent to $x^2 + 11x + 24$ additively and $(x + 3)(x + 8)$ multiplicatively.

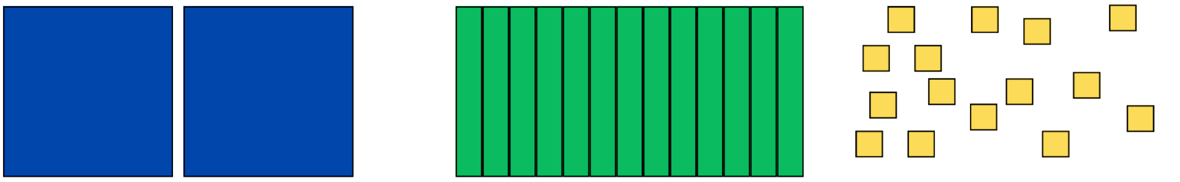


Note: One square (x^2 tile), eleven planks (x tiles) and 24 unit tiles arranged additively and multiplicatively.

4.2 Transforming all Trinomials to a Monic Quadratic

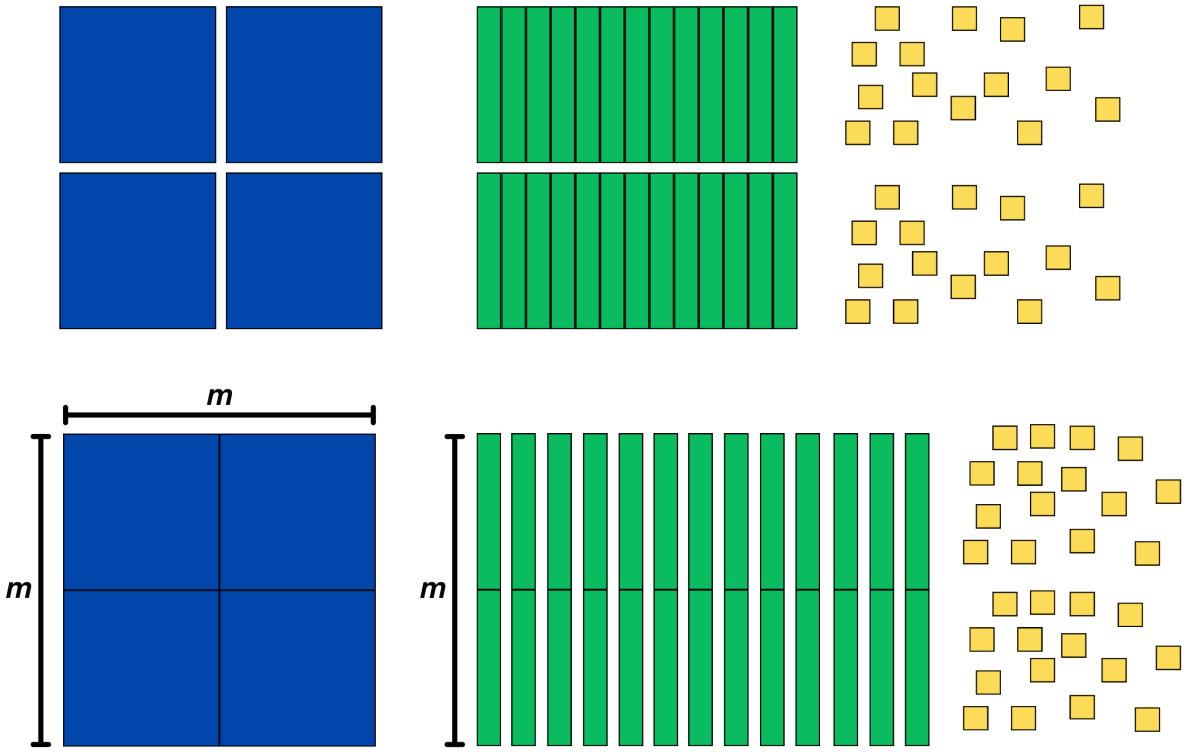
The key to this transformation is converting any non-monic quadratics into one with a leading coefficient of one. In terms of algebra tiles, this is equivalent to taking a quadratic with multiple x^2 tiles and creating a single square out of these pieces. For example, in the quadratic $2x^2 + 13x + 15$ illustrated in Figure 6, we need to transform the two blue tiles into a single square.

Figure 6: Collection of algebra tiles representing/equivalent to $2x^2 + 13x + 15$.



One way to create a single square is to divide all the blocks in half. However, since there are 13 planks and 15 units, this creates non-integer amounts of tiles. Alternatively, a single square can be created by doubling the number of tiles and reconceptualizing the x-tiles as a single m-tile whose dimensions are twice as long (see Figure 7). Symbolically, the doubling creates $2(2x^2 + 13x + 15) = 4x^2 + 26x + 30$, which through the substitution $m = 2x$, converts $(2x)^2 + 13(2x) + 30$ to $m^2 + 13m + 30$.

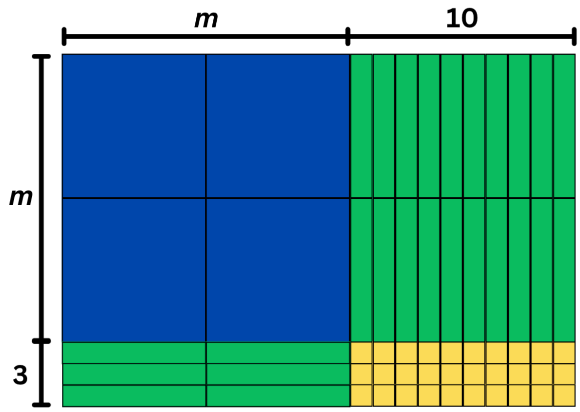
Figure 7: Reconceptualizing two x-tiles as one m-tile.



4.3 Familiar Factoring

Having transformed the previous challenging quadratic into a familiar monic one, we can use the previously generalized pattern to factor. We simply have to arrange the 30 blocks into a rectangle whose dimensions add up to 13. Again, symbolically, $m^2 + 13m + 30 = (m + 10)(m + 3)$.

Figure 8: Using a previously generalized pattern to factor.



4.4 Inversion

Finally, we invert back to our previous problem, by removing the half of the tiles we added and reconceptualizing the blocks with dimension x. Symbolically, we convert back to x and distribute the $\frac{1}{2}$: $\frac{1}{2}(m + 10)(m + 3) = \frac{1}{2}(2x + 10)(2x + 3) = (x + 5)(2x + 3)$.

Figure 9: Visualizing the inversion process by removing half of the tiles.

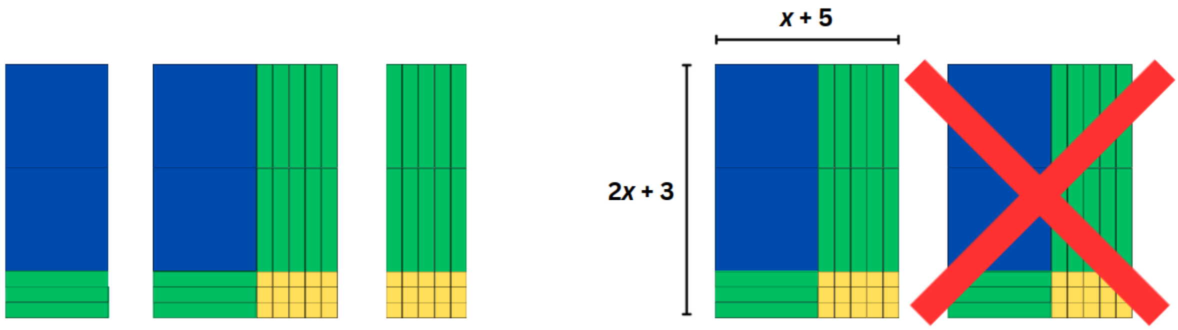
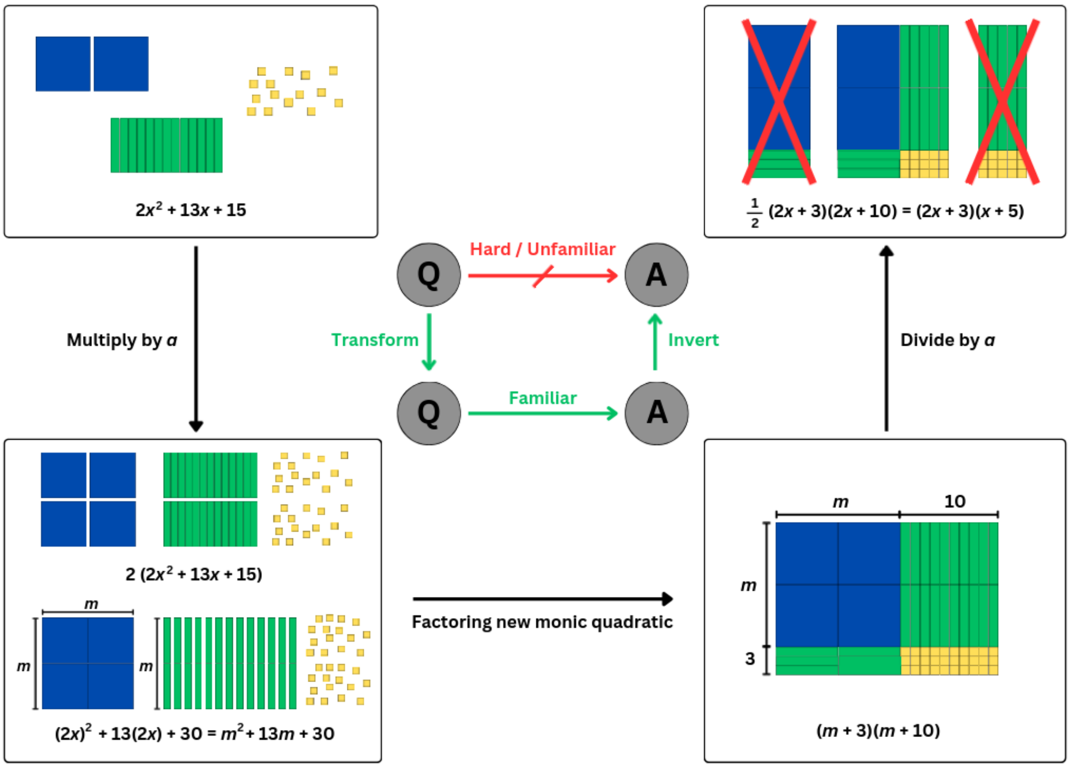


Figure 10 illustrates the process of transforming an unfamiliar factoring task into a familiar one using algebra tiles.

Figure 10: The transformational process applied to factoring.



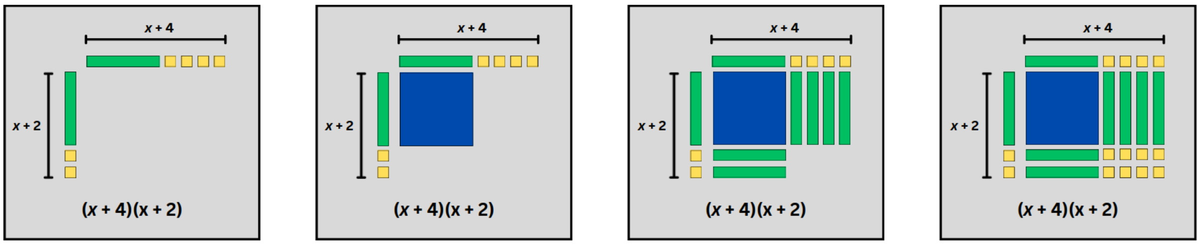
5 Instructional Trajectory

As highlighted earlier, algebra tiles provide a concrete representation that illustrate the relationship between the additive and multiplicative structures associated with quadratics as well as the transformational process that creates simpler quadratics with a leading coefficient of 1. However, while algebra tiles can be used to demonstrate such relationships, we believe they can also serve as a powerful tool to support students in making such connections on their own. Below we articulate a possible instructional sequence that positions students to do just that.

5.1 Multiplying with Algebra Tiles

Before engaging students in exploring factoring, we encourage teachers to introduce algebra tiles through multiplication by first showing students how an area model can be used to represent the product of two binomials. The process of multiplication is more direct than factoring and familiarizes students with how the various terms of the polynomials can be configured to form a rectangle. As such, the key is to emphasize the meaning of the pieces and how they fit together. For example, as illustrated by Figure 11 (Lischka & Stephens, 2020), the multiplication of $(x + 2)(x + 4)$ can be represented using an area model by creating a rectangle of width $(x + 4)$ and height $(x + 2)$. The quadratic partial product term results in a blue x^2 tile which is positioned in the upper left. Alongside this square are the linear terms of $4x$ and $2x$ represented by the six x by 1 planks. Finally, the eight units complete the rectangle in the bottom right corner. These different components illustrate the expanded expression $x^2 + 4x + 2x + 8 = x^2 + 6x + 8$, which results from distributing $(x + 2)(x + 4)$. Once students have such understanding in place, they are then positioned to work in the opposite direction and explore factoring by rearranging tiles and looking for patterns for how to organize them into a rectangular array.

Figure 11: Multiplying binomials using algebra tiles (Lischka & Stephens, 2020).

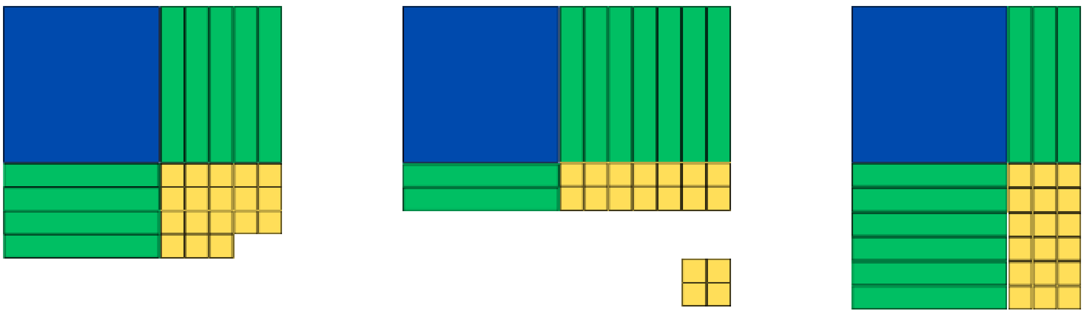


5.2 Factoring with Algebra Tiles

While algebra tiles provide an accessible, tactile approach to factoring quadratics, we emphasize that the goal is not simply for students to be able to arrange the different pieces to form a rectangular array, but to develop an understanding of the underlying pattern for how to do so. Working with teachers and students alike, we have noticed two approaches to solving a problem like $x^2 + 9x + 18$: (a) starting by arranging the 9 planks on each side of the x^2 square in different configurations in an attempt to leave a rectangle of 18 blocks or (b) working with different rectangular arrays of the 18 blocks and seeing which one, if any, results in the distribution of 9 planks adjacent to the square.

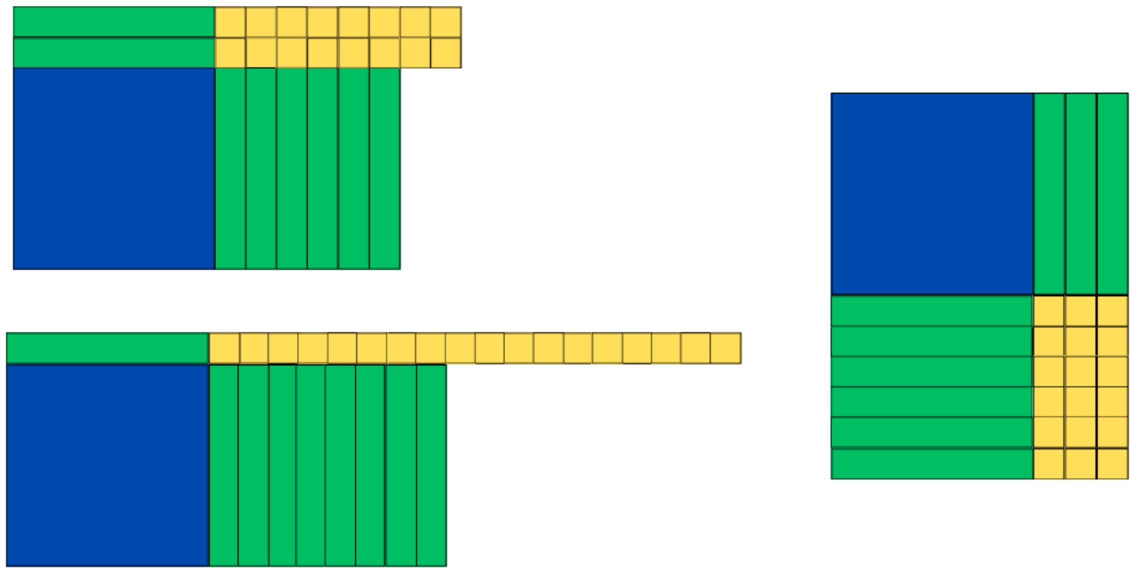
Notably, while the latter method of starting with blocks aligns with the common symbolic method of guess and check, most students tend to explore with the planks first. For example, they might begin by placing 4 planks below and the remaining 5 to the side of the square. This leaves a 5×4 region that must be filled with the remaining 18 units (see Figure 12). Since $5 \times 4 > 18$, placing the 18 units here would not complete the rectangle, leaving 2 spaces unaccounted for. Similarly, they could try by distributing 2 and 7 planks on each side of the square, forming a 2 by 7 rectangular region, which the 18 units would fill but leave 4 unused. Eventually, students find that we need a distribution of the 9 planks so that the two numbers have a sum of 9 and a product, the dimensions of remaining rectangle, of 18. The correct configuration for this problem would be $3 + 6 = 9$ planks and $3 \times 6 = 18$ units.

Figure 12: Possible distributions of 9 planks.



Alternatively, students can start by making rectangles with the blocks and seeing what combination, if any, results in a distribution of 9 planks. With 18 units, there are 3 options for rectangles that can be created. For example, students might start with a 2×9 rectangle and notice that this orientation requires more planks than available. Similarly, if they choose a 1×18 orientation, this results in a distribution of 1 and 18 planks, again requiring more planks than available. Eventually, through questioning, students realize that the dimensions of the rectangular blocks must add up to the number of planks (See Figure 13).

Figure 13: Possible rectangular distributions of 18 units.



While the above example illustrates how to arrange algebra tiles for a polynomial that factors, we have also found that asking students to explore and grapple with problems that do not factor is particularly helpful to understand the relationship between the number of planks and units. In contrast to a factorable case where students see the solution as the end of the task, the tension of not being able to do so encourages students to investigate and explain why not. As they do, they are forced to generalize a method for knowing if a problem works, supporting them in concluding that the number of planks, which is the coefficient b, needs to match the sum of the dimensions ($l \times w$) of the rectangular array formed by the units which is c.

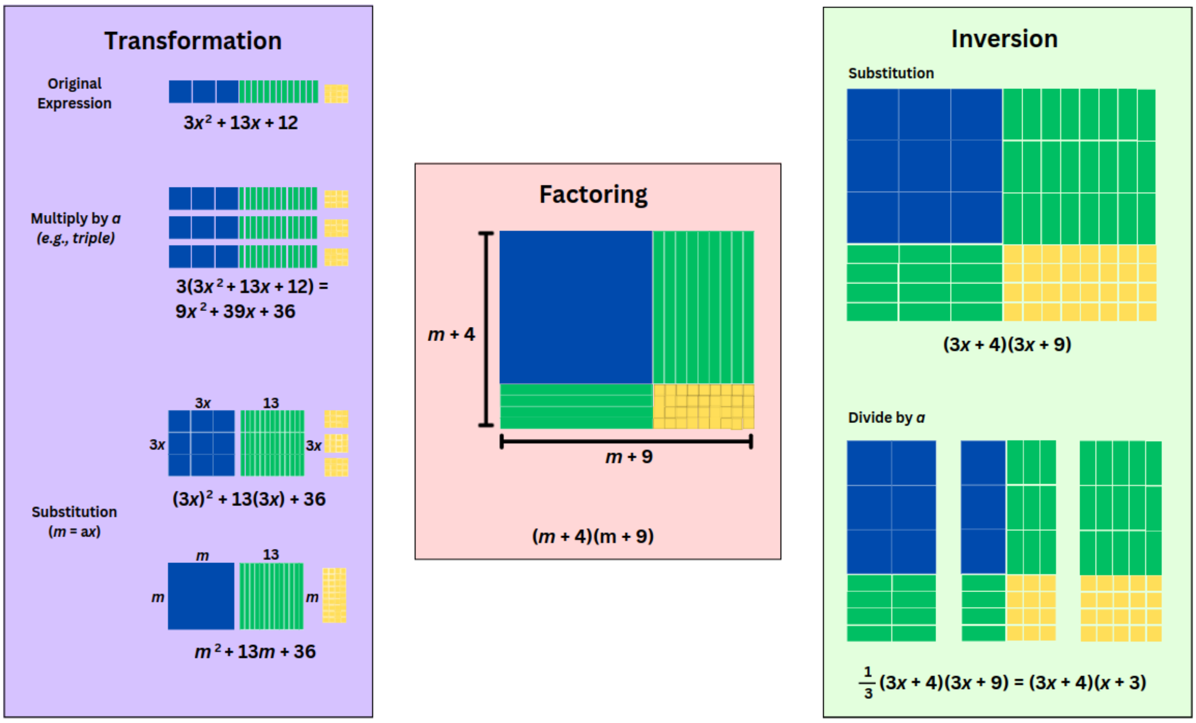
5.3 Introducing the Transformational Approach to Factoring: The Unfamiliar Case when $a \neq 1$

Before introducing the transformational approach to factoring, we encourage teachers to begin by asking students to use algebra tiles to factor non-monic quadratics. Once again, the purpose is to set a tone of exploration, challenging students to come up with a generalized pattern similar to the one discovered for monic quadratics. Inevitably, students will be able to form a rectangle with the tiles on individual problems, but not identify a systematic method to do so. As such, the point is for students to see the challenges associated with such work and to motivate the need for an easier method.

While much of the instruction of the transformational approach will inevitably be more direct, we encourage teachers to illustrate and familiarize students with the method using algebra tiles before introducing the steps symbolically. These steps will be discussed later once students are fluent carrying out the transformation using the tiles. Introduced too early, symbolic manipulation can become a meaningless set of rules, not too different from other algorithms. By foregrounding the concrete objects, students become familiar with how the transformational process creates the more familiar structure with a leading coefficient of 1 (i.e., a single x^2 square). Once they have internalized this process with the blocks, then the symbolic representations serve to simply capture their previous understanding of the different steps.

Below we have created a table outlining the various steps to transform the quadratic $3x^2 + 13x+12$ using both algebra tiles and the associated symbolic representations. When students are ready to embed their understanding of the process into symbolic form, we imagine teachers walking through and discussing each step with the tiles (this can also be done with digital manipulatives at https://mathsbot.com/manipulatives/tiles) and then asking students to represent the transformation symbolically. Basically, students would fill out the final column. We anticipate the substitution step being the most challenging and have subsequently broken this up into multiple steps. In particular, we encourage teachers to first label the tiles using $3x$ and ask students to write this out as an algebraic expression. This step helps students conceptualize $3x$ as a single symbolic unit, supporting them when the substitution, $m = 3x$, is introduced. Foregrounding the transformation process with algebra tiles and then connecting the two representations provides more meaning for what the symbolic steps represent, specifically why we multiply by a (to create a square) and what substitution means (change dimensions to a 1×1 square).

Table 1. *Steps to transform the quadratic* $3x^2 + 13 + 12$.



We acknowledge that factoring quadratics is difficult for students. However, rather than providing students black box algorithms consisting of a sequence of operations where the rationale is hidden, we advocate for leveraging this opportunity to help students understand how mathematicians overcome such challenges by transforming the problem into a simpler version with a known solution. Such an approach not only supports them in developing a method to factor quadratics, but fosters the mathematical practice of looking for and making use of structure. Moreover, by using algebra tiles to illustrate this method, students can see and carry out the transformation without relying on symbolic manipulation that often conceals the meaning.

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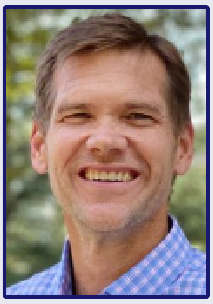

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