

Silent Geometry, Talking Roots

Hervé Svoboda and Valentin Roustan

Lycée Amiral de Grasse, Provence-Alpes-Côte d'Azur, France

Abstract

In the classroom, a recurring question often arises: “Are we allowed to do this with square roots – and what about that?” This article presents three wordless geometric proofs that help students visualize both the rules that are allowed and those that are not when working with square roots. The proofs treat multiplication ($\sqrt{ab} = \sqrt{a}\sqrt{b}$), division ($\sqrt{a/b} = \sqrt{a}/\sqrt{b}$), and the additive inequality ($\sqrt{a+b} < \sqrt{a} + \sqrt{b}$), relying only on the definition of a square root, the notion of geometric scaling, and – in the third case – the Pythagorean theorem and the triangle inequality. A short classroom vignette and a Note for Teachers are included to support implementation.

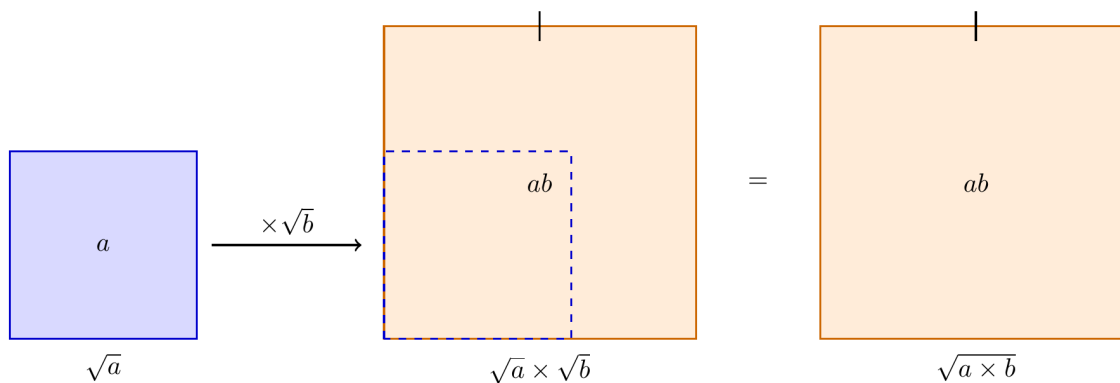
Keywords: Square Roots, Visual Proof, Wordless Proof, Geometric Reasoning, Secondary Mathematics

1 Three Visual Proofs

Throughout this section, we assume that a and b are two strictly positive real numbers.

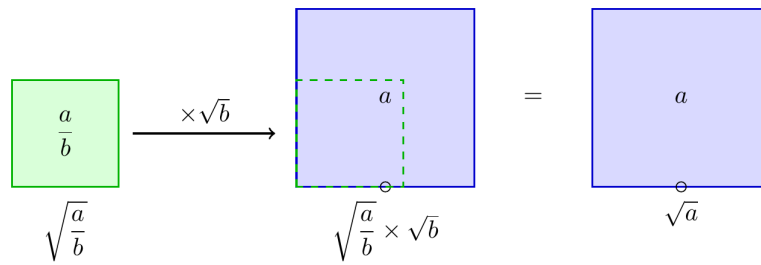
Rule 1: $\sqrt{a \times b} = \sqrt{a} \times \sqrt{b}$

Proof.



Rule 2: $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$

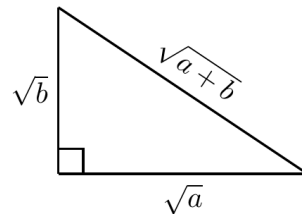
Proof.



$$\sqrt{\frac{a}{b}} \times \sqrt{b} = \sqrt{a} \implies \sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$$

Rule 3: $\sqrt{a+b} < \sqrt{a} + \sqrt{b}$ ($\implies \sqrt{a+b} \neq \sqrt{a} + \sqrt{b}$)

Proof.



2 A Classroom Vignette

The following exchange suggests one way Rule 1 can unfold in class. A teacher (T) is at the board with the figure for $\sqrt{a \times b} = \sqrt{a} \sqrt{b}$; a student (S) has just been asked what she notices.

T: I am going to start with this small blue square. Its area is a . What is the length of its side?

S: \sqrt{a} .

T: Good. Now I stretch the square — I scale it by a factor of \sqrt{b} — so each side becomes $\sqrt{a} \times \sqrt{b}$. What is the area of the new square?

S: $(\sqrt{a} \times \sqrt{b})^2$, since the area of a square is the side squared.

T: Right. And we also know that the area went from a to ab , because we scaled each side by \sqrt{b} , so the area was multiplied by b . Compare those two expressions for the new area.

S: $(\sqrt{a} \times \sqrt{b})^2 = ab$.

T: Which means the side, $\sqrt{a} \times \sqrt{b}$, has to equal \sqrt{ab} . We did not need the rules of exponents to get there — we just looked at a square.

This kind of Socratic exchange tends to surface the algebraic rule as a *consequence* of the visualization, rather than as a fact to be memorized.

3 Note for Teachers

These activities are suitable for students in lower secondary school onward — approximately ages 13–16, corresponding to the French *cycle 4* and early *lycée*, or to U.S. grades 8–10. Prerequisites are

modest: students should be comfortable with the definition of a square root and with the idea of geometric scaling. Rule 3 additionally requires familiarity with the Pythagorean theorem and the triangle inequality.

It is traditional to prove the three previous properties using algebra, the rules of exponents, and the properties of the square function. In this article we have chosen instead to present three wordless geometric proofs, simple, elegant, and accessible through visualization rather than algebraic manipulation.

- For the first two rules, we rely only on the definition of the square root and on the notion of geometric scaling.
- For the last, the Pythagorean theorem and the triangle inequality allow us to see almost at a glance why the inequality holds.

In words, then, the three figures of Section 1 establish the following. *Rule 1.* Scaling a square of side \sqrt{a} by a factor of \sqrt{b} produces a square of area ab ; its side, $\sqrt{a} \times \sqrt{b}$, must therefore equal $\sqrt{a \times b}$. *Rule 2.* Scaling a square of side $\sqrt{a/b}$ by a factor of \sqrt{b} produces a square of area a ; the resulting side, $\sqrt{a/b} \times \sqrt{b}$, must equal \sqrt{a} . *Rule 3.* By the Pythagorean theorem, the hypotenuse of a right triangle with legs \sqrt{a} and \sqrt{b} has length $\sqrt{a+b}$; by the triangle inequality, this hypotenuse is strictly less than the sum of the legs, $\sqrt{a} + \sqrt{b}$.

The goal is not to replace the classical demonstration, but to offer an intuitive entry point: the figures depict squares, ratios, and transformations that concretely illustrate the operations involving square roots.

The literature on student understanding of mathematical proof suggests that purely formal demonstrations often fail to produce conviction. Tall and Vinner's (1981) classic study of concept image and concept definition, and subsequent work by Hanna (1990) on the distinction between proofs that *prove* and proofs that *explain*, document the same phenomenon: students may correctly carry out an algebraic derivation yet remain unconvinced of the underlying mathematical truth. Visual proofs of the kind presented here often supply the missing conviction — the figure shows *why* the identity must hold, not merely that it follows from a manipulation of symbols.

These wordless proofs can also inspire broader reflection on the teaching of algebraic identities and on the role of visualization in mathematics — a role that we believe is fundamental to deep understanding. They encourage a shift in learning posture, one in which understanding begins by “seeing why” before “calculating how.”

All three properties are part of the French secondary mathematics curriculum, and analogues appear in essentially every national curriculum at the same level. We hope that this visual and geometric approach will enrich classroom discussions and help students develop a stronger conceptual understanding, grounded in reasoning through images rather than through formulas alone.

Acknowledgement

The authors wish to thank the editor and the anonymous reviewer for their constructive feedback on an earlier version of this manuscript.

About the Authors



Hervé Svoboda is a senior secondary school teacher (*agrégé de mathématiques*) and has been teaching since 2010 at Lycée Amiral de Grasse in southeastern France. Hervé is passionate about visual mathematics and “proofs without words,” which he integrates into his teaching and mathematical explorations. His work focuses on making abstract concepts more accessible through imagery and intuition.



Valentin Roustan is an *agrégé* mathematics teacher at Lycée Amiral de Grasse, with a longstanding passion for the intuition and beauty behind “proofs without words” in mathematics. He is also a content creator on social media under the pseudonym “mathelps,” where he is followed by nearly 200,000 people.

References

- Hanna, G. (1990). Some pedagogical aspects of proof. *Interchange*, 21(1), 6–13. <https://doi.org/10.1007/BF01809605>
- Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics*, 12(2), 151–169. <https://doi.org/10.1007/BF00305619>