The Interval (0,1) as Cosmic Archive: A Classroom Story*

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Abstract

This paper shares a teaching experience from a basic mathematics course at Farhangian University, Iran, across three cohorts (100 student-teachers). Starting with concrete counting (a shepherd and his pebbles), the course moved through infinite sets (Hilbert's Hotel, countability arguments) to binary representation. At the end, students realized that a finite text—including a human life story—can be encoded, under a fixed canonical convention, as a binary fraction corresponding to a unique point in the interval $(0, 1)$. This moment spontaneously raised metaphysical questions among students about destiny, free will, and the nature of reality. This is a classroom story, shared in hopes that other teachers might find it interesting or thought-provoking.

Keywords: teaching infinity, Hilbert's Hotel, binary representation, mathematics metaphor

1 Why I Am Writing This

I am a mathematics teacher educator at Farhangian University in Gonabad, Iran. I teach basic mathematics to first-year student-teachers who will one day teach in elementary schools. Most of them come to my class thinking mathematics is a collection of rules, formulas, and right answers. They are often anxious, sometimes resistant, and rarely expecting wonder.

This paper is my attempt to share something that happened in my classroom over several semesters. Something I did not plan entirely. Something that surprised me as much as it surprised my students.

I am not writing this as a researcher. I have no data to show, no control group, no statistical significance. I am writing as a teacher who watched his students' eyes light up when they realized that the interval $(0, 1)$ — that simple, unassuming stretch of number line — could be seen as containing every story ever told or yet to be told.

If you are a teacher, especially a teacher of mathematics, I hope this story gives you ideas or simply reminds you why you chose this profession.

2 The Setting

2.1 Where and Who

The course is Basic Mathematics for first-year elementary education students at Farhangian University (Shahid Beheshti Campus, Gonabad, Iran). Students enter directly from secondary school and come

A note to readers: This is simply a classroom story. I am not making research claims, testing a hypothesis, or proving that my approach works better than others. I am sharing an experience that mattered to me and to my students. If you find something useful here for your own teaching, I am glad. If not, I hope at least you enjoy the journey.

from three streams: Mathematics, Experimental Sciences, and Humanities. Their mathematical backgrounds vary widely.

I taught this course across three cohorts over two academic years: 24 undergraduate students and 33 associate-degree students in Fall 2024, and 43 undergraduate students in Fall 2025.

The instructional sequence was the same for all cohorts: four 90-minute sessions on set theory, number systems, and infinity.

2.2 What This Paper Is and Is Not

- **Is:** A reflective classroom narrative
- **Is:** A sharing of what I did and what I observed
- **Is:** An invitation for other teachers to try similar approaches
- **Is not:** An empirical research study
- **Is not:** A claim that my method is better than others
- **Is not:** A proof that students learned more or better

The quotations from students are reconstructions from notes I wrote after class. I have done my best to be faithful to what was said, but these are not transcripts. They are meant to give a sense of the classroom atmosphere, not to serve as data.

2.3 Teacher's Evolving Practice

Teaching the same sequence across three cohorts (2024–2025) allowed me to refine my practice with each iteration. By the third cohort, the pacing felt more natural, student resistance was anticipated and addressed earlier, and the metaphysical questions emerged more spontaneously. This paper thus documents not a perfected lesson but an evolving practice—one that improved through repeated engagement with the same pedagogical challenges.

The four sessions unfold as a gradual journey, beginning with familiar questions and representations and leading, step by step, toward a perspective on $(0, 1)$ that is far less familiar.

3 Part One: The Shepherd and the Pebbles[1]

3.1 An Ancient Story

I began the first session with a story:

> *A shepherd, thousands of years ago, faced a problem. Each evening, he needed to know if all his sheep had returned from the pasture. He could not count — not in the way we do. So he devised a method. Each morning, as a sheep left the pen, he placed a pebble in a leather pouch. One sheep, one pebble. In the evening, as each sheep returned, he removed one pebble. If no pebbles remained, all sheep were home. If pebbles remained, some were missing.*

This story, which I did not invent but inherited from the mathematical tradition, contains the seed of one of mathematics' most powerful ideas: one-to-one correspondence.

[1]The Latin word for pebble is *calculus*—the same root behind *calculate* and *calculation*—so the shepherd's stones already carry the etymology of computation.

3.2 From Pebbles to Bijection

I then introduced the formal definition:

Definition 3.1 (One-to-One Correspondence)**.** A **one-to-one correspondence** between two sets A and B is a pairing that assigns to each element of A exactly one element of B, and to each element of B exactly one element of A. If such a pairing exists, we say that A and B have the same **cardinality** (or size), and we write $|A| = |B|$.

Definition 3.2 (Injective, Surjective, and Bijective Functions)**.** A function $f\colon A \to B$ is called *injective* (one-to-one) if

$$f(x_1) = f(x_2) \Longrightarrow x_1 = x_2 \qquad \text{for all } x_1, x_2 \in A.$$

It is called *surjective* (onto) if

$$\forall y \in B,\ \exists x \in A \text{ such that } f(x) = y.$$

If f is both injective and surjective, it is called *bijective*. A bijective function establishes a one-to-one correspondence between the elements of A and B.

We use $\mathbb{N} = \{1, 2, 3, \ldots\}$ for the natural numbers, $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$ for the integers, $\mathbb{Q}$ for the rational numbers (numbers of the form a/b with $a, b \in \mathbb{Z}$ and $b \neq 0$), $\mathbb{R}$ for the real numbers (the complete ordered field, identified with the points on the real line), $\mathbb{R}^2 = \{(x, y) \mid x, y \in \mathbb{R}\}$ for the two-dimensional Euclidean space, $\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) \mid x_i \in \mathbb{R},\ 1 \leq i \leq n\}$ for the n-dimensional Euclidean space, $\mathbb{E}$ for the even integers, $\mathbb{O}$ for the odd integers, and $(0, 1)^2$ for the Cartesian product $(0, 1) \times (0, 1)$, the open unit square.

The shepherd never needed to know the number of sheep. He only needed the correspondence. This, I told my students, is the heart of the mathematical way of thinking: before counting comes matching.

3.3 First Shocks

We then looked at infinite sets.

Example 3.3 (Even numbers and natural numbers)**.** The function $f(n) = 2n$ pairs each natural number n with an even number $2n$. This is a bijection. So the set of even numbers has the same cardinality as the set of all natural numbers.

Example 3.4 (Odd numbers and natural numbers)**.** $g(n) = 2n - 1$ gives a bijection between $\mathbb{N}$ and the odd numbers.

Example 3.5 (Integers and natural numbers)**.**

$$g(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ -\frac{n-1}{2} & \text{if } n \text{ is odd} \end{cases}$$

gives a bijection between $\mathbb{N}$ and $\mathbb{Z}$, establishing $|\mathbb{Z}| = |\mathbb{N}|$.

The objections came quickly.

"But the even numbers are only half of the natural numbers!"

"The part cannot equal the whole."

These objections are exactly right — for finite sets. But infinite sets break this rule. In fact, the ability to be mapped onto a proper subset of itself is the defining characteristic of an infinite set.

I did not dismiss their objections. I validated them: "You are feeling exactly what mathematicians felt when Cantor first proposed these ideas. Leopold Kronecker, a very respected mathematician, rejected Cantor's work publicly (Weber, 1893). Your intuition is just trained on finite cases." Then I invited them to follow the logic where it led. "Let us be shepherds," I said. "Can we establish a one-to-one correspondence between natural numbers and integers?" The students agreed, by saying yes or nodding, that such a correspondence exists. "So the two sets must be the same size," I said.

The silence that followed was profound. The logic was undeniable, yet the conclusion remained deeply unsettling. This tension, between what we feel must be true and what logic demonstrates to be true, is the crucible in which mathematical understanding is forged.

One student expressed this resistance in practical terms: *"Professor, these concepts have no practical use. I will be teaching elementary school children. Why do I need to learn this? It has no application for me."*

My response invited patience and trust in the journey:

> *Be patient and try to understand the material well. It is true that you will work with young children, but children often misuse terms like 'infinite', 'countable' and 'countless.' I expect you, as future teachers, to understand these concepts correctly. Furthermore, we will reach points where I am confident your perspective on mathematics will shift. Your beliefs about mathematics will influence your teaching. I want your attitude toward mathematics to change—or at least for you to encounter other dimensions of it.*

Remark 3.6 (The Defining Property of Infinity)**.** The Euclidean axiom that "the whole is greater than the part" holds for finite sets but fails for infinite sets. In fact, the ability to be put into one-to-one correspondence with a proper subset of itself is a *defining characteristic* of infinite sets. An infinite set is precisely one that can be mapped bijectively onto a proper subset of itself.

Definition 3.7 (Countably Infinite Set)**.** A set A is **countably infinite** if there exists a one-to-one correspondence between A and the set of natural numbers $\mathbb{N} = \{1, 2, 3, \ldots\}$. In other words, we can list the elements of A as a sequence $a_1, a_2, a_3, \ldots$ without repetition or omission. The cardinality of such a set is denoted by $\aleph_0$ (aleph-null).

So,

$$|\mathbb{E}| = |\mathbb{O}| = |\mathbb{N}| = |\mathbb{Z}| = \aleph_0.$$

3.4 Rationals: The Next Surprise

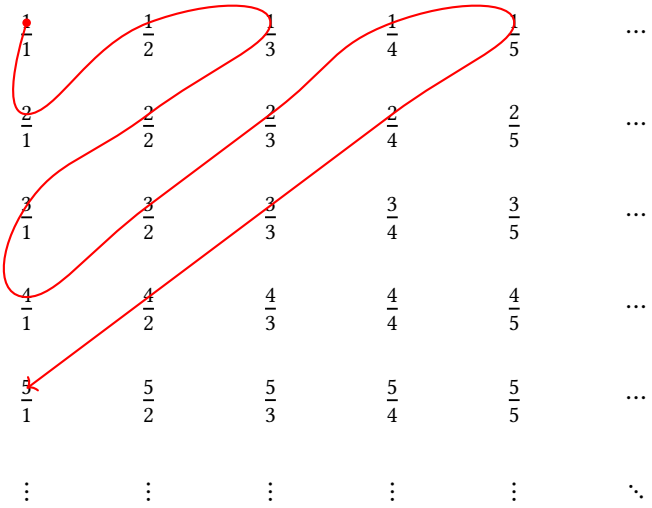
Before presenting Cantor's ingenious enumeration of the rationals, I paused to clarify a key logical point with my students. A set being countably infinite means, by definition, that there exists a bijection between that set and the natural numbers. But a bijection is nothing more than a systematic way to *list* the elements of the set as a sequence $a_1, a_2, a_3, \ldots$ without repetition or omission. In other words, to prove that a set is countably infinite, it suffices to provide an explicit **enumeration** of its elements. Conversely, if such an enumeration exists, the set is countably infinite. This equivalence between "having a bijection with $\mathbb{N}$" and "being listable as a sequence" is the conceptual bridge that makes Cantor's diagonal argument and his enumeration of the rationals so powerful.

We then turned to the rational numbers $\mathbb{Q}$. Between any two integers lie infinitely many fractions. Surely this set must be larger than $\mathbb{N}$?

But Cantor showed otherwise. Using his diagonal enumeration of an infinite grid, we can list all positive rationals in a sequence. Including negatives and zero, the entire set $\mathbb{Q}$ is countable.

Example 3.8 (Rationals are Countably Infinite)**.** We can arrange all positive rational numbers in an infinite grid:

Figure 1
Cantor's Diagonal Enumeration of the Positive Rationals



Note. Following the arrows through the array reaches every positive rational exactly once, which is what it means for the set to be countable. Duplicates such as $\frac{2}{2}$ are skipped.

By traversing this array along diagonals, a path that goes $\frac{1}{1}$, $\frac{1}{2}$, $\frac{2}{1}$, $\frac{3}{1}$, $\frac{2}{2}$ (skipping duplicates like $\frac{2}{2}$ which equals $\frac{1}{1}$), $\frac{1}{3}$, $\frac{1}{4}$, $\frac{2}{3}$, etc., we can assign a natural number to each rational. The enumeration above lists only positive rationals. To include negative rationals and zero, we modify the list as follows: start with 0, then interleave each positive rational with its negative. That is, from the enumeration of positive rationals $p_1, p_2, p_3, \ldots$, construct a new enumeration $0, p_1, -p_1, p_2, -p_2, p_3, -p_3, \ldots$. This new list contains every rational number exactly once, proving that $\mathbb{Q}$ (all rationals) is countably infinite (Cantor, 1932; Dauben, 1990).

Remark 3.9. The countability of the rationals is one of mathematics' most beautiful surprises (Körner, 1996). It reveals that density (the property of having elements between any two others) is not the same as cardinality. The rationals are dense in the real line, yet they are only countably infinite.

3.5 The Countable Union of Countable Sets

Before examining the relationship between rationals and irrationals, we need a key fact about countable sets.

Proposition 3.10 (Union of Two Countable Sets is Countable)**.** If A and B are countable sets, then their union $A \cup B$ is also countable.

Proof. Since A and B are countable, we can list their elements as sequences $A = \{a_1, a_2, a_3, \ldots\}$ and $B = \{b_1, b_2, b_3, \ldots\}$ (allowing finite lists if either set is finite). Now list the elements of $A \cup B$ by interleaving: $a_1, b_1, a_2, b_2, a_3, b_3, \ldots$, skipping any duplicates. This produces an enumeration of $A \cup B$, proving it is countable. □

Remark 3.11. The same idea extends to any finite union of countable sets. Moreover, with a more sophisticated diagonal argument (similar to Cantor's enumeration of the rationals), one can prove that a *countably infinite* union of countable sets is also countable. For our purposes, the finite case suffices.

3.6 The Uncountable: Entering the Darkness

Having explored the countable infinite, we turned to the question that would shatter our complacency: Are *all* infinite sets the same size? Cantor's most revolutionary discovery was that they are not.

Theorem 3.12 (Cantor's Theorem)**.** The set of real numbers in the interval $(0, 1)$ is **not** countable. It has a strictly larger cardinality than $\mathbb{N}$.

Diagonal Argument. Suppose, for contradiction, that $(0, 1)$ is countable. Then we could list all its elements as a sequence:

$$\begin{aligned} r_1 &= 0.a_{11}a_{12}a_{13}a_{14}\ldots \\ r_2 &= 0.a_{21}a_{22}a_{23}a_{24}\ldots \\ r_3 &= 0.a_{31}a_{32}a_{33}a_{34}\ldots \\ &\vdots \end{aligned}$$

where each a_{ij} is a digit from 0 to 9. To avoid ambiguity, we choose the representation that does not end in infinite 9s for numbers with two decimal representations (e.g., we use $0.2000\ldots$ rather than $0.1999\ldots$).

Now construct a new number $r = 0.b_1b_2b_3b_4\ldots$ by defining:

$$b_n = \begin{cases} 1 & \text{if } a_{nn} \neq 1 \\ 2 & \text{if } a_{nn} = 1 \end{cases}$$

This number r is clearly in $(0, 1)$, but it differs from r_1 in the first decimal place, from r_2 in the second decimal place, and in general from r_n in the n-th decimal place. Therefore, r is not in the list. This contradicts our assumption that the list contained all numbers in $(0, 1)$. Hence, $(0, 1)$ is uncountable. □

The moment of revelation in the classroom was electric. Students who had just accepted that the rationals are countable—dense though they be—now faced a far more radical truth: there exist infinities so vast that they cannot be listed, cannot be enumerated, cannot be tamed by any sequential process.

Definition 3.13 (Uncountable Set and the Continuum)**.** A set that is neither finite nor countably infinite is called **uncountable**. The cardinality of $(0, 1)$ is called the **cardinality of the continuum** and is denoted by $\mathfrak{c}$ (or $2^{\aleph_0}$). This cardinality is strictly greater than $\aleph_0$ (Cantor, 1932; Dauben, 1990).[2]

Having established that $(0, 1)$ has cardinality $\mathfrak{c}$, we extended the discussion to other sets of the same size.

Example 3.14 (The Real Line $\mathbb{R}$ Has the Same Cardinality as $(0, 1)$)**.** The function $f : (0, 1) \to \mathbb{R}$ defined by $f(x) = \tan\left(\pi x - \frac{\pi}{2}\right)$ is a bijection. It maps the open interval $(0, 1)$ continuously onto the entire real line. Hence, $|\mathbb{R}| = \mathfrak{c}$.

[2]While the cardinality of the continuum, $\mathfrak{c}$, is strictly greater than $\aleph_0$ (Cantor's theorem), identifying it specifically with $\aleph_1$—the first uncountable cardinal—is the Continuum Hypothesis (CH). As Gödel (1940) and Cohen (1963) demonstrated, CH is independent of the standard axioms of set theory (ZFC). We therefore state only that $\mathfrak{c} > \aleph_0$, leaving the precise value of $\mathfrak{c}$ open.

Example 3.15 (The Plane $\mathbb{R}^2$ Has the Same Cardinality as $\mathbb{R}$)**.** At first glance, the plane seems far larger than the line. However, Cantor showed that $|\mathbb{R}^2| = \mathfrak{c}$ as well. To see this, consider any point in the unit square (x, y) with decimal expansions $x = 0.x_1x_2x_3\ldots$ and $y = 0.y_1y_2y_3\ldots$ (choosing the non-terminating representation to avoid ambiguity). Map this point to the number $z = 0.x_1y_1x_2y_2x_3y_3\ldots$ in $(0, 1)$. Cantor showed that a bijection between the unit square $(0, 1)^2$ and the interval $(0, 1)$ exists; one intuitive approach is digit interleaving, although a rigorous proof requires additional care. Since $(0, 1)$ has cardinality $\mathfrak{c}$ and $(0, 1)^2$ has the same cardinality as $\mathbb{R}^2$, we conclude $|\mathbb{R}^2| = \mathfrak{c}$.

These examples reveal a stunning fact: the line, the plane, and even higher-dimensional spaces $\mathbb{R}^n$ all have the same cardinality $\mathfrak{c}$. The uncountable infinity of the continuum is, in a sense, a single "size" that encompasses many familiar geometric objects.

3.7 Rationals and Irrationals: Stars and Darkness

We then examined the relationship between rationals and irrationals within $(0, 1)$. The rationals in this interval are countable—a consequence of the countability of all rationals. The irrationals, however, tell a different story.

Proposition 3.16. The set $(\mathbb{R} \setminus \mathbb{Q}) \cap (0, 1)$ of irrational numbers in $(0, 1)$ is uncountable.

Proof. We know that $(0, 1) = (\mathbb{Q} \cap (0, 1)) \cup ((\mathbb{R} \setminus \mathbb{Q}) \cap (0, 1))$. The left-hand side is uncountable, while $\mathbb{Q} \cap (0, 1)$ is countable. If $(\mathbb{R} \setminus \mathbb{Q}) \cap (0, 1)$ were also countable, their union would be countable (since the union of two countable sets is countable). This contradiction shows that $(\mathbb{R} \setminus \mathbb{Q}) \cap (0, 1)$ must be uncountable. □

We reflected on this beautiful analogy: the rationals are like stars in the night sky—infinitely many, yet isolated points—while the irrationals are like the vast, dark space between them, immeasurably larger. This image made the uncountable tangible, and the students smiled with quiet satisfaction.

4 Hilbert's Hotel: A Window into the Infinite

4.1 The Hotel and Its Paradoxes

The German mathematician David Hilbert, building on Cantor's work, created a thought experiment that perfectly captures the strange behavior of infinite sets (Hilbert, 1926; Wallace, 2003). I presented it to the class with theatrical flair:

> *Imagine a hotel with infinitely many rooms, numbered 1, 2, 3, ... ad infinitum. The rooms are all occupied—every room has a guest. Now a new guest arrives and asks for a room. Can the hotel accommodate this new guest without asking anyone to leave?*

The initial response was unanimous: "Impossible. The hotel is full."

"But," I responded, "let us think like mathematicians, not hotel managers. We can ask the guest in room 1 to move to room 2, the guest in room 2 to move to room 3, and so on—each guest moves to the room with the next number. What happens?"

A moment of silence, then dawning realization. Room 1 becomes vacant. The new guest can be accommodated. The hotel, though "full," has space for one more.

4.2 Scaling the Paradox

I pressed further: "Now suppose a bus arrives with 40 new guests. Can we accommodate them?"

Some students immediately saw the pattern: "Move each existing guest from room n to room n+40, and rooms 1 through 40 become vacant."

"Excellent. Now suppose a bus arrives with infinitely many new guests—countably infinitely many. Can we accommodate them?"

This question produced the most animated discussion of the entire course. Some students insisted it was impossible—infinity plus infinity must surely overflow any hotel, like adding water to an already full glass. Others attempted solutions: "Move each guest to room $n + \infty$" (which they quickly realized was meaningless). Still others sat in contemplative silence, sensing that a trick was coming.

When the solution was revealed, several students exclaimed, *"Wow, my God, there's room again!"* Another said, *"I never thought we could fit that many passengers into the hotel."* A third added, *"I wish I had thought of that!"*

> *Ask each existing guest to move from room n to room 2n. Then all odd-numbered rooms become vacant. Since there are infinitely many odd numbers, we have rooms for all the new guests.*

For the student-teachers, this was more than an abstract exercise. They were experiencing what it means to think mathematically—following logic even when it contradicts intuition. This experience, I suggested, would serve them well when teaching their own students.

4.3 Hilbert's Hotel and the Prime Numbers

To push the paradox to its limit, I introduced a final scenario:

Example 4.1 (Infinitely Many Buses, Each with Infinitely Many Passengers)**.** Suppose not one, but countably infinitely many buses arrive, each carrying countably infinitely many new guests. Can Hilbert's Hotel accommodate them all?

The solution requires a more sophisticated strategy using prime numbers. Let r be the guest's current room number and s be the passenger's seat number:

1. Assign each current guest to room 2^r. This places them in rooms that are powers of 2.
2. Assign passengers on the first bus to rooms 3^s.
3. Assign passengers on the second bus to rooms 5^s.
4. In general, assign passengers on the k-th bus to rooms p_k^s, where p_k is the k-th prime number.

By the Fundamental Theorem of Arithmetic, every positive integer has a unique prime factorization. Therefore, no two guests—whether original or new—can be assigned the same room. All are accommodated.

Remark 4.2. This strategy works precisely because we are dealing with *countably infinite* sets. The natural numbers, the room numbers, and the bus seat numbers all have the same cardinality $\aleph_0$. The primes provide a systematic way to map the product of two countable infinities (the buses and their seats) onto a single countable infinity (the rooms) (Wallace, 2003).

The students were stunned. The infinite, which seemed so simple when we first encountered it, had revealed itself to be infinitely more complex and wondrous than they had ever imagined.

As the solution sank in, whispers spread across the room. "Oh my God!" one student exclaimed. Another said, "I never thought we could fit that many passengers into the hotel." A third added, "Wow, whose idea was this?" Then a student sighed, "I wish I had thought of that!"

5 The World of Base-2

5.1 Why Base-10?

In the following session, I shifted focus from the vastness of infinity to the intimacy of number systems. I began with a deceptively simple question: "Why do we count in groups of ten?"

The answers were predictable: "Because we have ten fingers." This led to a discussion of how number bases are conventions, not mathematical necessities. Ten is special only because of our biology. Any integer $b > 1$ can serve as a base, and different bases reveal different aspects of numbers.

Definition 5.1 (Base-b Representation)**.** Let $b > 1$ be an integer. Any positive integer N can be uniquely written as:

$$N = a_k b^k + a_{k-1} b^{k-1} + \cdots + a_1 b^1 + a_0 b^0$$

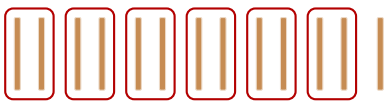
where each a_i is an integer satisfying $0 \leq a_i < b$, and $a_k \neq 0$. This is called the **base-b representation** of N, and we write $N = (a_k a_{k-1} \ldots a_1 a_0)_b$.

5.2 Building Base-2 from Scratch

To make the concept tangible, I presented a hands-on activity that had become a tradition in my class. Taking 13 matchsticks, I asked students to group them in pairs:

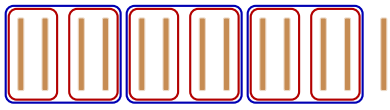
13 matchsticks

13 matchsticks. How many pairs? Six pairs, with one left over. So our 'ones' place (2^0) is 1.



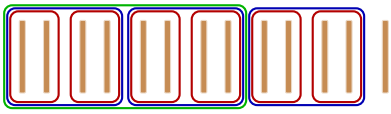
6 pairs + 1 left over

Now group the six pairs into groups of two pairs each—groups of four. From six pairs, we get three groups of four, with zero pairs left over. So our 'twos' place (2^1) is 0.



3 groups of four, no pairs left over

Group the three groups of four into groups of two fours—groups of eight. We get one group of eight, with one group of four left over. So our 'fours' place (2^2) is 1.



One group of eight

We now have one group of eight, and cannot form a group of sixteen. So our 'eights' place (2^3) is 1.

Reading from the highest place down, we have $13 = (1101)_2$.

5.3 The Division Algorithm

I then showed the more efficient method that would serve for any base: successive division.

Example 5.2 (Converting 13 to Binary by Division)**.**

$$13 \div 2 = 6 \text{ remainder } 1 \quad (a_0 = 1)$$
$$6 \div 2 = 3 \text{ remainder } 0 \quad (a_1 = 0)$$
$$3 \div 2 = 1 \text{ remainder } 1 \quad (a_2 = 1)$$
$$1 \div 2 = 0 \text{ remainder } 1 \quad (a_3 = 1)$$

Reading the remainders from bottom to top gives $(1101)_2$.

Example 5.3 (Converting Binary to Decimal)**.** Convert $(1101)_2$ to decimal:

$$(1101)_2 = 1 \times 2^3 + 1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0 = 8 + 4 + 0 + 1 = 13$$

5.4 Binary Fractions and Infinite Series

Just as we can have decimal fractions, we can have binary fractions:

$$(0.1)_2 = \frac{1}{2} = 0.5$$
$$(0.01)_2 = \frac{1}{4} = 0.25$$
$$(0.001)_2 = \frac{1}{8} = 0.125$$

Then came the moment that connected binary to our earlier understanding of infinite series and limits.

Example 5.4 (The Repeating Binary Fraction)**.** Consider $(0.111111\ldots)_2$. This represents the infinite series:

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$$

This is an infinite geometric series with first term $a = \frac{1}{2}$ and ratio $r = \frac{1}{2}$. Its sum is:

$$S = \frac{a}{1-r} = \frac{\frac{1}{2}}{1-\frac{1}{2}} = \frac{\frac{1}{2}}{\frac{1}{2}} = 1$$

Thus, $(0.111111\ldots)_2 = 1$ exactly. This is the binary analog of $0.999\ldots = 1$ in decimal.

The students were fascinated by this parallel. The infinite, which had seemed so foreign in the context of set theory, now appeared in the familiar guise of fractions and series.

5.5 Applications of Binary Numbers

The students were further intrigued by the practical applications of base-2:

The Magic Trick. Cards with numbers 1-31 can be prepared based on binary representation. Each card corresponds to a power of 2.

1	3	5	7
9	11	13	15
17	19	21	23
25	27	29	31

2	3	6	7
10	11	14	15
18	19	22	23
26	27	30	31

4	5	6	7
12	13	14	15
20	21	22	23
28	29	30	31

8	9	10	11
12	13	14	15
24	25	26	27
28	29	30	31

16	17	18	19
20	21	22	23
24	25	26	27
28	29	30	31

By noting which cards contain a chosen number, the number is revealed—the sum of the first numbers on those cards. The students were delighted to realize that they were, in effect, performing binary conversion without knowing it.

Beyond its mathematical elegance, I found the trick especially valuable pedagogically. Its element of mystery can turn routine number practice into an engaging mathematical experience. Students are often surprised that the chosen number can be recovered simply by stacking the indicated cards—even without looking at them. At the same time, the performer must remember the first number on each indicated card and add them mentally, quietly practicing working memory and mental arithmetic. In this way, students move from watching a trick to actively carrying out the mathematics themselves: the rigor is there, but it is concealed within wonder.

Computer Arithmetic. All digital computers use binary representation for data and arithmetic. The 0s and 1s they had been manipulating were the very language of modern technology. Understanding how these simple digits combine to perform complex calculations reveals the elegant logic underlying every digital device.

Binary arithmetic derives its elegance and efficiency from the simplicity of its operations. Addition and multiplication tables reduce to trivial rules, making hardware implementation remarkably straightforward.

Subtraction is typically implemented via two's complement representation, where the negative of a number n is represented as $2^N - n$ for an N-bit system. This transforms subtraction $m - n$ into addition $m + (2^N - n)$, eliminating the need for dedicated subtraction circuitry. Any final carry beyond N bits is discarded, yielding the correct result.

Example 5.5 (Subtraction via Two's Complement)**.** Consider subtracting $(0111)_2$ (7) from $(1010)_2$ (10) in a 4-bit system ($N = 4$, so $2^4 = 16$):

$$\begin{aligned} m = (1010)_2 &= 10 \\ n = (0111)_2 &= 7 \\ \text{Two's complement of } n: \quad 2^4 - n = 16 - 7 = 9 &= (1001)_2 \end{aligned}$$

Now perform addition instead of subtraction:

$$\begin{array}{rcccccc} & 1 & 0 & 1 & 0 & (10) \\ + & 1 & 0 & 0 & 1 & (9) \\ \hline 1 & 0 & 0 & 1 & 1 & \\ \uparrow & & & & & \end{array}$$

Discarding the final carry (indicated by the upward arrow) leaves $(0011)_2 = 3$, which is exactly $10 - 7$. This elegant mechanism means the computer needs only addition hardware; subtraction is handled by first taking the two's complement and then adding.

Remark 5.6 (The Elegance of Binary Arithmetic)**.** The simplicity of binary arithmetic—where multiplication tables shrink to the trivial facts $0 \times 0 = 0$, $0 \times 1 = 0$, and $1 \times 1 = 1$—is precisely what makes digital computers possible. A machine that need only distinguish between two states (on/off, high voltage/low voltage) can implement these rules with astonishing reliability and speed.

Digital Data Representation. Text, images, sound, and video are all encoded as sequences of bits. Every word you read on a screen, every image you see, every sound you hear is ultimately a string of 0s and 1s. Understanding how these diverse forms of information are translated into binary reveals the remarkable versatility of this simple representation.

Text. The most straightforward encoding. Standards like ASCII (American Standard Code for Information Interchange) assign each character a unique 7-bit code, allowing $2^7 = 128$ characters—sufficient for English letters, digits, and punctuation. For example, the letter 'A' is $(1000001)_2 = 65$, while 'a' is $(1100001)_2 = 97$. Unicode (The Unicode Consortium, 2022), the modern standard, uses up to 32 bits per character, accommodating over a million glyphs covering virtually all writing systems on Earth—from Persian script to Chinese ideograms.

Images. A digital image is a rectangular grid of *pixels* (picture elements). Each pixel's color is represented by a binary code. In the common RGB (Red-Green-Blue) model, 24 bits are used per pixel—8 bits for each color channel—allowing $2^{24} \approx 16.7$ million distinct colors. A full HD image (1920×1080 pixels) thus contains nearly 2 million pixels, each requiring 24 bits, totaling about 6 megabytes of raw data. Compression algorithms like JPEG (International Telecommunication Union, 1992) exploit patterns in this binary data to reduce file sizes dramatically.

Sound. Sound waves are continuous analog signals. To digitize them, we measure (*sample*) the wave's amplitude thousands of times per second and record each measurement as a binary number. This process is governed by the Nyquist-Shannon sampling theorem (Nyquist, 1928; Shannon, 1948, 1949), which states that to accurately capture a frequency, the sampling rate must be at least twice that frequency. CD-quality audio, designed to capture the full range of human hearing (up to 20 kHz), samples at 44.1 kHz with 16-bit resolution (Philips & Sony Corporation, 1980; Watkinson, 1994). For stereo sound, requiring two independent channels, the bit rate is calculated as:

$$44,100 \frac{\text{samples}}{\text{second}} \times 16 \frac{\text{bits}}{\text{sample}} \times 2 \text{ channels} = 1,411,200 \text{ bits per second}$$

This means a single minute of uncompressed CD-quality stereo audio (PCM format) requires over 10.5 megabytes of raw data ($1,411,200 \times 60 \div 8 = 10,584,000$ bytes). For comparison, compressed formats like MP3 achieve bit rates of 128-320 kbps by applying perceptual coding—removing audio information that is less audible to human ears. Despite this compression, the precise binary representation of the original samples theoretically allows perfect reproduction of the original sound when played back.

Video. Digital video combines image sequences with synchronized audio, creating the most data-intensive of all media. A single second of 4K video at 30 frames per second represents billions of bits (Society of Motion Picture and Television Engineers, 2012). Compression standards like H.264 (International Telecommunication Union, 2021) achieve remarkable efficiency by encoding only the *differences* between consecutive frames rather than each frame independently—an elegant application of mathematical redundancy reduction.

This universality of binary representation means that the same fundamental operations—the addition, multiplication, and logical operations discussed above—can manipulate text, images, sound, and video with equal ease. A computer processor makes no distinction between an instruction, a number, a letter, or a pixel; all are simply patterns of bits to be processed according to programmed rules. For student-teachers, this insight reveals how mathematics dissolves the boundaries between seemingly disparate domains, providing a unified language for the entire digital world.

Remark 5.7. The remarkable fact is that, depending on the processor's speed, the handling and interpretation of this stream of zeros and ones occurs at an astonishingly rapid pace. Billions upon billions of such operations are executed each second, and from this ceaseless binary dance emerges the entire digital universe. Every pixel rendered on a screen, every packet routed across the internet, every neural network trained in an artificial intelligence system ultimately reduces to the manipulation of binary digits according to these fundamental rules.

For student-teachers, understanding this foundational layer demystifies modern technology and reveals mathematics as the hidden language powering the digital age. The entire information age rests upon this simple yet profound foundation.

As the students explored these ideas, excited whispers filled the room. "Wow, how interesting! I never thought mathematics could be so useful!" one exclaimed. Another said, "How beautiful!" A third student reflected, "I used to play this card trick as a child, but I never knew how it worked. Now I understand it much better." Then a student added, "So mathematics is everywhere!" Another concluded, "What fascinating applications!"

6 Part Four: From Binary to The Cosmic Archive

"Life is good for only two things, discovering mathematics and teaching mathematics."

— Siméon-Denis Poisson

6.1 Encoding Life Stories

At this point, all the pieces were in place for the culminating insight of the course. The room grew quiet as I posed the thought experiment I had been building toward for weeks:

> *Imagine that the story of your life—every event, every conversation, every thought you've ever had—could be typed out as a text. This text, in a computer, would be stored as a sequence of bits: 0s and 1s. This sequence, though long, would be finite.*

I let this sink in, then continued:

"Now place '0.' before this binary sequence. What do you have?"

"A number in base-2 between 0 and 1," came the whispered response.

"Exactly. The story of your life corresponds to a unique point in the interval (0,1)."

6.2 A Mathematical Theorem

What followed was not merely a poetic speculation, but a precise mathematical statement that we could prove together.

Definition 6.1 (Life Story Encoding)**.** For this construction, we represent a human life story by a finite text written according to a fixed canonical convention—for example, fixed rules for spacing, capitalization, punctuation, and a fixed allowable character set consisting of standard ASCII characters with nonzero ASCII codes. We then use a fixed-width ASCII encoding, representing each character by its standard ASCII code in an 8-bit form. Thus, each finite text determines a finite binary sequence

$$S = s_1 s_2 \dots s_n, \qquad s_i \in \{0, 1\}.$$

We regard this binary sequence as the encoding of the story.

Proposition 6.2 (A Unique Point for Each Canonical Text)**.** Let $S = s_1 s_2 \dots s_n$ be the binary sequence obtained from a nonempty canonical text under the fixed-width ASCII encoding specified in Definition 6.1. Define

$$x = \sum_{i=1}^{n} \frac{s_i}{2^i} = 0.s_1 s_2 \dots s_{n2}.$$

Then $x \in (0, 1)$, and the map from canonical texts to points in $(0, 1)$ is one-to-one.

Proof. Since the text is nonempty and every character allowed by our canonical text convention has a nonzero ASCII code, its binary encoding contains at least one 1. Hence

$$x = \sum_{i=1}^{n} \frac{s_i}{2^i} > 0.$$

Also,

$$x \leq \sum_{i=1}^{n} \frac{1}{2^i} < 1.$$

Thus $x \in (0, 1)$.

It remains to show uniqueness. Suppose two canonical texts produce the same number x. Their finite binary representations can represent the same real number only if one binary sequence is obtained from the other by appending zeros. But, under our fixed-width ASCII encoding, the final character of a nonempty text is represented by a nonzero 8-bit code, so its encoding cannot end in an additional block of zeros corresponding to an appended character. Therefore the two binary sequences must be identical. Since the encoding is fixed, the canonical texts themselves must be identical. Hence the map is one-to-one. □

The proposition was simple, almost trivial in its mathematics. But its implications were staggering.

6.3 The Library of All Lives

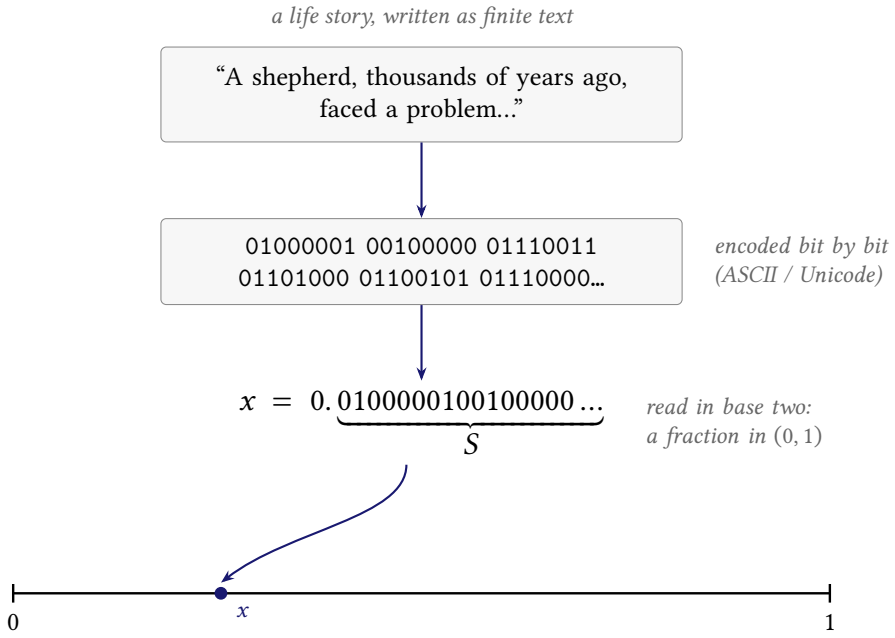
I extended the thought, letting the implications unfold:

"The story of my life, of every student in this room, of every human being who has ever lived or will ever live—each, if written, becomes a binary sequence. Each, with '0.' prefixed, becomes a point in (0,1). Every life, every destiny, corresponds to a number in this interval."

The room grew deathly quiet. Students were processing something far beyond ordinary mathematics.

"And consider: all the stories that have ever been told, all the novels ever written, all the histories recorded, all the conversations yet to occur—every possible narrative that can be expressed in language—each can be encoded as binary and mapped to a point in this interval."[3]

Figure 2
Encoding a Finite Text as a Single Point in the Interval $(0, 1)$



Note. Any finite text encodes to a finite string of bits, and prefixing "0." reads that string as a binary fraction — a single point in the interval $(0, 1)$. Nothing here is schematic: the displayed bits are the ASCII encoding of the sentence's opening characters ("A shep..."), and the displayed 16-bit prefix, 0.0100000100100000_2, has the exact value $\frac{521}{2048} = 0.25439453125$.

Remark 6.3 (The Interval as Cosmic Archive)**.** The interval $(0, 1)$ can represent encodings of:

- Every possible life story
- Every story ever told or yet to be told
- Every novel ever written
- Every conversation ever had
- Every possible narrative that language can express

It is a mathematical library of all possible human experiences—a cosmic archive beyond any physical library, beyond any digital storage, beyond any conceivable technology.

6.4 The Students' Response

The reaction was unforgettable. Students who had been struggling with abstract concepts suddenly saw mathematics in an entirely new light. The interval $(0, 1)$ was no longer just a set of numbers; it had become a cosmic library, a mathematical echo of something far greater than itself.

One student reflected: "If that is true, then my entire life story is already a point in the interval (0,1). So my whole life is already written there. That means I have no freedom to change it."

[3]Some students connected this metaphor to religious concepts familiar in their tradition. For example, for some Muslim students, this vision immediately evoked the concept of the Preserved Tablet (Lawh Mahfuz) mentioned in the Quran (85:21-22): the eternal record in which all that has happened and all that will happen is inscribed. The interval (0,1) thus served as a powerful mathematical metaphor for this spiritual concept (*The Holy Qur'an*, 1934).

I responded: "No. With every decision you make, you are actually living one of the many different possible stories. The fact that a story can be encoded as a point does not mean it is predetermined. You are completely free."[4]

Another student asked: "When I make decisions, which version of my life story is written there?"

I replied: "All possible stories can be represented there. You simply choose which one to live. In fact, a person is not just themselves and the life they have lived. They are themselves and all the lives they could have lived." Another student reflected, with a tone of awe:

> *I have never seen a library so small yet so full of books!*

A different student added:

> *Professor, I have never had such a view of mathematics before. I have never looked at the world from this angle.*

7 Reflections on Mathematics and Metaphysics

7.1 Mathematics as Bridge

What this teaching experience revealed, I believe, is the profound capacity of mathematics to serve as a bridge between domains we usually keep separate: the concrete and the abstract, the finite and the infinite, the human and the divine. The student-teachers did not merely learn definitions and algorithms; they encountered wonder.

Mathematics, often presented as a cold, mechanical discipline of formulas and computations, became for them a source of metaphysical reflection. The counterintuitive nature of infinite sets challenged their assumptions about part and whole. Hilbert's Hotel made them grapple with the logic of infinity in a tangible way. Base-2 encoding connected the most practical of technologies (computers) to the most eternal of questions (destiny).

7.2 The Role of the Teacher

As teacher-educators, our task is not merely to convey content but to create experiences that transform how our students think. The student-teachers in this class will themselves become mathematics teachers. What will they take from this experience?

I hope they will take the understanding that mathematics is not a closed book of answers but an open door to questions. That the most profound insights often come from following logic where it leads, even when it contradicts intuition. That mathematics can speak to the deepest human concerns—the nature of infinity, the relationship between part and whole, the possibility of encoding existence itself.

Most of all, I hope they will remember the moment of silence in the classroom when the connection was made—the moment when mathematics ceased to be a subject and became a window onto the infinite.

7.3 Limitations and Cautions

I must also note the limitations of this approach. The connection between binary encoding and the cosmic archive is metaphorical, not literal. It does not prove any metaphysical doctrine, nor does it

[4] I also acknowledged the depth of the question: "This touches on deep philosophical debates about determinism and free will — debates that mathematics alone cannot resolve. The mathematical encoding shows that all possible stories can be represented as distinct points in (0,1). Which point corresponds to the life you actually live depends on choices that mathematics does not prescribe. The map is not the territory."

reduce spiritual truths to mathematical formulas. Rather, it uses mathematical concepts to illuminate and enrich our understanding of traditional ideas.

Furthermore, such an approach requires sensitivity to students' diverse backgrounds and beliefs. The essential point—that mathematics can open windows to metaphysical reflection—remains valid across cultural contexts.

While the metaphor of the interval $(0, 1)$ as a "cosmic archive" proved pedagogically powerful, it must be interpreted with care. The encoding of life narratives as binary sequences is a representational construct rather than an ontological claim. Mathematical existence does not imply causal determination, nor does it collapse distinctions between possibility and actuality.

The purpose of the metaphor is educational: to illuminate the expressive capacity of mathematical structures and to stimulate reflective engagement, not to advance doctrinal assertions.

8 Pedagogical Implications for Teacher Education

The observed classroom experience suggests several concrete implications for mathematics teacher education. Based on what I learned across three cohorts, I offer the following recommendations for teacher educators who wish to cultivate wonder alongside rigor.

1. **Start with a paradox, not a definition.** Before introducing the formal definition of countability, begin with an experience that challenges intuition. Hilbert's Hotel or the shepherd's pebbles work well. Let students sit with the cognitive conflict: "How can a hotel with no empty rooms accommodate a new guest?" Only after they have struggled—and perhaps argued—should the instructor introduce the language of bijections and cardinality. The definition then becomes a resolution, not an imposition.
2. **Use narrative and metaphor as cognitive scaffolds.** Abstract concepts like infinite cardinalities or binary encoding become tangible when embedded in stories. The shepherd counting sheep, the infinite hotel, the stars and darkness of the irrationals—these images linger long after formulas are forgotten. Teacher educators should model this narrative approach explicitly, so that future teachers internalize it as a pedagogical tool for their own classrooms.
3. **Invite metaphysical questions explicitly.** After teaching binary representation, do not shy away from questions like: "If every text can be encoded as a number, what does that mean for stories not yet written?" or "Does the interval (0,1) contain every possible life?" Such questions transform mathematics from a collection of techniques into a living inquiry. My experience showed that students not only engaged with these questions but spontaneously connected them to destiny, free will, and the nature of reality.
4. **Document and honor student wonder.** Keep a notebook of student reactions—surprise, hesitation, awe, spontaneous connections. These moments are not distractions from learning; they are evidence that learning has become meaningful. Sharing these documented reactions with future teachers (e.g., "Last semester, a student exclaimed, 'Wow, my God, there's room again!' ") normalizes wonder and encourages them to listen for similar moments in their own future classrooms.
5. **Repeat and refine across cohorts.** The same instructional design yielded consistent patterns across three independent cohorts over two academic years. This repetition not only strengthened trustworthiness but also allowed me to refine my responses to common student questions. Teacher educators should view each iteration of a course not as a repetition but as an opportunity to test, adjust, and deepen the pedagogical design.

6. **Embrace productive cognitive conflict.** When a student objects, "But the even numbers are only half of the natural numbers!" do not correct them immediately. Sit with the objection. Validate it. Then gently invite the class to follow the logic of one-to-one correspondence. The tension between intuition and logic is not a problem to be eliminated; it is the crucible in which mathematical understanding is forged. Teacher educators should train future teachers to recognize such moments as opportunities, not obstacles.

These recommendations support the view that effective mathematics education for teachers must address not only procedural competence but also philosophical orientation toward the subject. When future teachers experience mathematics as a source of wonder and metaphysical reflection, they are more likely to create similar experiences for their own students.

9 Conclusion: The Eternal Project

We began with a shepherd and his pebbles—the most primitive of counting methods, born of practical necessity. We moved through the paradoxes of Hilbert's Hotel, where the logic of infinity shattered our finite intuitions. We explored the elegant simplicity of binary numbers, discovering that every text, every story, every life can be encoded as a string of 0s and 1s. And we arrived at a vision of the interval (0,1) as a cosmic archive containing all possible human destinies.

This journey, undertaken over several class sessions with student-teachers at Farhangian University, demonstrates the power of mathematics to evoke wonder and inspire metaphysical reflection. It shows that mathematics is not merely a tool for calculation, but a language for exploring the deepest questions of existence.

The student-teachers who experienced this journey will carry with them not just definitions and theorems but a sense of mathematics as a living, breathing inquiry into the nature of reality. When they themselves teach, they will have the opportunity to pass on not just techniques but wonder—the wonder that comes from realizing that the universe is comprehensible, that infinity can be thought, and that the human mind, finite though it is, can reach toward the eternal.

In the end, perhaps this is the deepest purpose of mathematical education: not to produce human calculators, but to nurture human beings who can think abstractly, reason rigorously, and wonder profoundly. The shepherd with his pebbles, Cantor with his transfinite numbers, Hilbert with his imaginary hotel—all were engaged in the same eternal human project: making sense of the infinite, the eternal, and the meaningful.

The journey continues. Every class is a new beginning. Every student is a new explorer. And every moment of wonder is a glimpse of the infinite that lies beyond all formulas and proofs.

In the end, every classroom is a Hilbert's Hotel—infinitely open to new ideas, new questions, and new wonders.

Acknowledgments

I thank the student-teachers of Farhangian University, Gonabad, Iran, whose questions and wonder made this experience meaningful. I also thank the mathematicians and philosophers whose work provided the foundation for this journey: Georg Cantor, who taught us to think the unthinkable; David Hilbert, who showed us the paradoxes that illuminate truth; and the countless anonymous teachers, from ancient shepherds to modern professors, who have passed on the torch of mathematical wonder.

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